Př. 1

$$\lim_{n\to\infty} n\cdot\frac{\sqrt[n]{n^{2n}+(2n)^n}}{\sqrt[n]{n^{3n}+(3n)^n}} = \lim_{n\to\infty} n\cdot\frac{\sqrt[n]{n^{2n}+2^n n^n}}{\sqrt[n]{n^{3n}+3^n n^n}} =$$

$$\overset{(i)}{=} \lim_{n\to\infty} n\cdot\frac{\sqrt[n]{n^{2n}\left(1+\left(\frac{2}{n}\right)^n\right)}}{\sqrt[n]{n^{3n}\left(1+\left(\frac{3}{n^2}\right)^n\right)}} = \lim_{n\to\infty} n\cdot\frac{n^2\cdot\sqrt[n]{1+\left(\frac{2}{n}\right)^n}}{n^3\cdot\sqrt[n]{1+\left(\frac{3}{n^2}\right)^n}} =$$

$$\overset{(ii)}{=} \lim_{n\to\infty}\frac{\sqrt[n]{1+\left(\frac{2}{n}\right)^n}}{\sqrt[n]{1+\left(\frac{3}{n^2}\right)^n}} \overset{VoAL}{=} \frac{\overset{(iii)}{\lim_{n\to\infty}}\sqrt[n]{1+\left(\frac{2}{n}\right)^n}}{\lim_{n\to\infty}\sqrt[n]{1+\left(\frac{3}{n^2}\right)^n}} = \frac{1}{1} = \underline{1}$$

- $\lim_{n\to\infty}\sqrt[n]{1+\left(\frac{2}{n}\right)^n} \overset{(iv)}{=} 1$ dle věty o 2 strážnících, neb

$$\underset{\substack{\downarrow n\to\infty\\ 1}}{1} \le \sqrt[n]{1+\left(\frac{2}{n}\right)^n} \le \sqrt[n]{1+2} = \underset{\substack{\downarrow n\to\infty\\ 1}}{\sqrt[n]{3}} \qquad \forall n\in\mathbb{N}$$

- $\lim_{n\to\infty}\sqrt[n]{1+\left(\frac{3}{n^2}\right)^n} \overset{(v)}{=} 1$ dle věty o 2 strážnících, neb

$$\underset{\substack{\downarrow n\to\infty\\ 1}}{1} \le \sqrt[n]{1+\left(\frac{3}{n^2}\right)^n} \le \sqrt[n]{1+3} \le \underset{\substack{\downarrow n\to\infty\\ 1}}{\sqrt[n]{4}} \qquad \forall n\in\mathbb{N}$$

BODOVÁNÍ: maximum: 12 b.

správná úprava až do (i) ... +2b

(ii) ... +1b

(iii) ... +2b — za absenci VoAL −2b.

(iv),(v) ... +7b. — chybné odhady... −3b.

chybné předpoklady... −3b.

numerická chyba... −2b.

závažnější chyby ... dle zvážení

Př. 2

$$\lim_{x\to\frac{\pi}{4}} \left(2\sin^2 x\right)^{\frac{1}{\sin x-\cos x}} \overset{(i)}{=} \lim_{x\to\frac{\pi}{4}} e^{\frac{1}{\sin x-\cos x}\log\left(2\sin^2 x\right)} \underset{\text{VoLSF}}{=} e^{\lim\limits_{x\to\frac{\pi}{4}}\frac{1}{\sin x-\cos x}\log\left(2\sin^2 x\right)} \underset{(vi)}{=} \underline{\underline{e^{\sqrt{2}}}}$$

$$\cdot \lim_{x\to\frac{\pi}{4}} \frac{1}{\sin x-\cos x}\log\left(2\sin^2 x\right) \overset{(ii)}{=} \lim_{x\to\frac{\pi}{4}} \frac{\log\left(2\sin^2 x\right)}{2\sin^2 x-1}\cdot\frac{2\sin^2 x-1}{\sin x-\cos x} \underset{(v)}{\overset{\text{VoAL}}{=}} 1\cdot\sqrt{2} = \underline{\underline{\sqrt{2}}}$$

$$\circ \lim_{x\to\frac{\pi}{4}} \frac{\log\left(2\sin^2 x\right)}{2\sin^2 x-1} = \lim_{y\to 1}\frac{\log y}{y-1} \overset{\text{zatl.}\ (iii)}{=} \underline{\underline{1}}$$

VoLSF(P)

$g(x)=2\sin^2 x$

$f(y)=\frac{\log y}{y-1}$

$\lim\limits_{x\to\frac{\pi}{4}} g(x)=1$, $g(x)$ ryze monotónní na $P(\frac{\pi}{4},\frac{\pi}{6}) \Rightarrow g(x)\neq 1$ na $P(\frac{\pi}{4},\frac{\pi}{6})$

$$\circ \lim_{x\to\frac{\pi}{4}} \frac{2\sin^2 x-1}{\sin x-\cos x} = \lim_{x\to\frac{\pi}{4}} \frac{\sin^2 x-\cos^2 x}{\sin x-\cos x} = \lim_{x\to\frac{\pi}{4}} \sin x+\cos x \overset{(iv)}{=} \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2} = \underline{\underline{\sqrt{2}}}$$

BODOVÁNÍ:

maximum: 13

správný postup až do (i) ... + 1b.

(ii) ... + 2b.

(iii) ... + 4b.

- chybné použití VoLSF -4b až -1b dle závažnosti
- neověření podmínky P ... -2b
- chybné ověření podm. P ... -1b

(iv) ... + 4b.

(v) ... + 1b.

(vi) ... + 1b.

chybí zmínka VoAL ... -2b. (max 1x)

numerická chyba ... -2b.

závažnější chyby ... dle uvážení

3. $f(x) = \operatorname{arctg}(x-1) \cdot \left| \operatorname{arctg}^2 x - \frac{\pi^2}{16} \right|$

f spoj. na D_f

$\lim_{x \to}$

$$f(x) = \operatorname{arctg}(x-1) \left| \left(\operatorname{arctg} x - \frac{\pi}{4}\right)\left(\operatorname{arctg} x + \frac{\pi}{4}\right) \right|$$

(i)
$$f(x) = \begin{cases} \operatorname{arctg}(x-1)\left(\operatorname{arctg}^2 x - \frac{\pi^2}{16}\right) & \text{pro } x \in (-\infty, -1] \cup [1, \infty) \\ \operatorname{arctg}(x-1)\left(\frac{\pi^2}{16} - \operatorname{arctg}^2 x\right) & \text{pro } x \in [-1, 1] \end{cases}$$

nebo ekvivalentní zápis

(ii)
$$\Rightarrow \boxed{f'(x) = \begin{cases} \frac{1}{1+(x-1)^2}\left(\operatorname{arctg}^2 x - \frac{\pi^2}{16}\right) + \operatorname{arctg}(x-1)\, 2\operatorname{arctg} x \cdot \frac{1}{1+x^2} & \text{pro } x \in (-\infty, -1) \cup (1, \infty) \\ \frac{1}{1+(x-1)^2}\left(\frac{\pi^2}{16} - \operatorname{arctg}^2 x\right) - \operatorname{arctg}(x-1)\, 2\operatorname{arctg} x \cdot \frac{1}{1+x^2} & \text{pro } x \in (-1, 1) \end{cases}}$$

! kul. závorky

$$f'_-(-1) \overset{V3.27}{=} \lim_{x \to -1-} f'(x) = \lim_{x \to -1-} \frac{1}{1+(x-1)^2}\left(\operatorname{arctg}^2 x - \frac{\pi^2}{16}\right) + \frac{\operatorname{arctg}(x-1)\, 2\operatorname{arctg} x}{1+x^2} \overset{*}{=} \underline{\frac{\pi}{4}\operatorname{arctg} 2}$$

$$f'_+(-1) \overset{V3.27}{=} \lim_{x \to -1+} f'(x) = \lim_{x \to -1+} \frac{1}{1+(x-1)^2}\left(\frac{\pi^2}{16} - \operatorname{arctg}^2 x\right) - \frac{\operatorname{arctg}(x-1)\, 2\operatorname{arctg} x}{1+x^2} = 0 - \frac{\pi}{4}\operatorname{arctg} 2 = \underline{-\frac{\pi}{4}\operatorname{arctg} 2}$$

$$f'_-(1) \overset{V3.27}{=} \lim_{x \to 1-} f'(x) = \dots = \underline{0}$$

$$f'_+(1) \overset{V3.27}{=} \lim_{x \to 1+} f'(x) = \dots = \underline{0} \quad \overset{(iv)}{\Rightarrow} \underline{f'(1) = 0}$$

(iii)

V3.27 můžeme použít, neb v -1 je f spoj. zprava (zleva) a $\lim_{x \to -1\pm} f'(x)$ existuje.
(lze také postupovat přímo z def. $f'_\pm(a)$)

$$* = \frac{1}{5} \cdot 0 + \operatorname{arctg}(-2) \cdot 2\operatorname{arctg}(-1) \cdot \frac{1}{2} = +\frac{\pi}{4}\operatorname{arctg} 2$$

BODOVÁNÍ:

(spoj.) ... +1b.
správný výpočet každé (i) ... +2b.
(ii) ... +$\frac{1}{2}$b. (chybné krajní body (intervaly) ... −1b. za každý chybný krajní bod)
(iii) ... +2b. za $f'_+(-1)$, $f'_-(-1)$, ($f'_\pm(1)$ a $f'_-(1)$ dohr.)
(iv) ... +1b.

12b.

numerická chyba ... −2b.
závažná chyba ... dle uvážení

h) $f(x) = x \cdot \exp\left(\frac{1}{x^2-2}\right)$

a) $f(-x) = -x \exp\left(\frac{1}{x^2-2}\right) = -f(x) \Rightarrow \underline{f \text{ lichá}}$ +1b.

$\underline{D_f = \mathbb{R} \setminus \{\pm\sqrt{2}\}}$ +1b.

b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x \cdot \exp\left(\underbrace{\frac{1}{x^2-2}}_{\to 0}\right) \overset{\text{VoLSF, VoAL}}{=} \infty \cdot 1 = \underline{\infty}$ +0,5b.

z lichosti $\lim_{x\to-\infty} f(x) = \underline{-\infty}$ +0,5b.

$\lim_{x\to\sqrt{2}+} x \exp\left(\underbrace{\frac{1}{x^2-2}}_{\to +\infty}\right) \overset{\text{VoLSF, VoAL}}{=} \sqrt{2} \cdot (+\infty) = \underline{+\infty}$ +0,5b.

$\lim_{x\to\sqrt{2}-} x \exp\left(\underbrace{\frac{1}{x^2-2}}_{\to -\infty}\right) \overset{\text{VoLSF, VoAL}}{=} \sqrt{2} \cdot \cancel{(+\infty)}^{\,0} = \cancel{-\infty}^{\,0}$ +0,5b.

z lichosti $\lim_{x\to-\sqrt{2}+} f(x) = \cancel{-\infty}_{\,0}$ +0,5b., $\lim_{x\to-\sqrt{2}-} f(x) = \underline{-\infty}$ +0,5b.

$\boxed{\text{BODOVÁNÍ:}}$

a) 2b. sudost/lichost, Definiční obor

b) 3b. limity v krajních bodech D_f

c) 6b. intervaly monotonie ~~body extrému~~ derivace (vč. jednost.)

d) 2b. asymptoty

num. chyba...

neověření předp. VoLSF −1b.

c) $f'(x) = \exp\left(\frac{1}{x^2-2}\right) + x\exp\left(\frac{1}{x^2-2}\right) \cdot \frac{0-2x}{(x^2-2)^2} = \exp\left(\frac{1}{x^2-2}\right)\left(1 - 2\left(\frac{x}{x^2-2}\right)^2\right) =$

$= \exp\left(\frac{1}{x^2-2}\right)\left(\frac{x^4-4x^2+4-2x^2}{x^4-4x^2+4}\right) = \underbrace{\overbrace{\exp\left(\frac{1}{x^2-2}\right)}^{>0} \frac{x^4-6x^2+4}{\underbrace{(x^2-2)^2}_{>0}}}_{\text{nebo ekviv. tvar}}$ na $\mathbb{R} \setminus \{\pm\sqrt{2}\}$

+2b.

$\exp\left(\frac{1}{x^2-2}\right) \cdot \left(1 - \frac{2x^2}{(x^2-2)^2}\right)$

$x^4 - 6x^2 + 4 > 0$

$D = 36 - 16 = 20$

$x^2_{1,2} = \frac{6 \pm 2\sqrt{5}}{2} = 3 \pm \sqrt{5} \Rightarrow x_{1,2,3,4} = \pm\sqrt{3 \pm \sqrt{5}}$

$-\sqrt{3+\sqrt{5}}$ | $-\sqrt{2}$ | $-\sqrt{3-\sqrt{5}}$ | 0 | $\sqrt{3-\sqrt{5}}$ | $\sqrt{2}$ | $\sqrt{3+\sqrt{5}}$

$+ \quad - \quad - \quad + \quad + \quad - \quad - \quad +$

$\Rightarrow f \nearrow$ na $(-\infty, -\sqrt{3+\sqrt{5}}), (-\sqrt{3-\sqrt{5}}, \sqrt{3-\sqrt{5}}), (\sqrt{3+\sqrt{5}}, \infty)$

(nebo uz. int. mimo $\pm\sqrt{2}, \pm\infty$) +4b.

$f \searrow$ na $(-\sqrt{3+\sqrt{5}}, -\sqrt{2}), (-\sqrt{2}, -\sqrt{3-\sqrt{5}}), (\sqrt{3-\sqrt{5}}, \sqrt{2}), (\sqrt{2}, \sqrt{3+\sqrt{5}})$

za chybný interval a krajní bod −1b.

lok. max. v $-\sqrt{3+\sqrt{5}}, \sqrt{3-\sqrt{5}}$

lok. min. v $-\sqrt{3-\sqrt{5}}, \sqrt{3+\sqrt{5}}$

d) $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{\frac{1}{x^2-2}} = 1$ +0,5b.

$\lim_{x\to\infty} x e^{\frac{1}{x^2-2}} - x = \lim_{x\to\infty} \frac{\left(e^{\frac{1}{x^2-2}} - 1\right)}{\frac{1}{x^2-2}} \cdot \frac{x}{x^2-2} = 0$ +½

z lichosti f má v ∞ a $-\infty$ asymptotu $y = x$ +0,5b.